The Impacts of Fast Electrons and Magnetic Neutrons

By J. F. CARLSON AND J. R. OPPENHEIMER
University of California, Berkeley, California

(Received July 18, 1932)

In this paper we consider the behavior of electrons with energy very large compared to their proper energy in their passage through matter, and further treat the impacts suffered by a certain type of hypothetical elementary neutral particle whose existence was tentatively suggested by Pauli. In the introduction we outline the problem and the methods to be employed, and give a summary of the formulae which embody our results. In Section I we develop the method suggested by Møller for the relativistic treatment of impacts; in (a) we apply it to the impacts of two free electrons; in (b) we show how it is to be applied to those impacts of a fast electron in which little energy is transferred to the secondary; in (c) we develop the theory of the magnetic neutron, and apply Møller's method to the treatment of its impacts. In II we give the detailed calculation of the energy transfers from a fast electron to the electrons of the matter through which it is passing, and compute the range and ionizing power of the primary electron. In III we apply the theory of the neutron to compute the number and nature of its impacts.

INTRODUCTION

RECENT experiments on the cosmic rays and on the penetrating radiation produced in the artificial disintegration of beryllium have raised again the question of the behavior of particles of very high energy in their passage through matter. Both these radiations have been shown to be extremely penetrating; and cloud chamber photographs have shown that both are accompanied by electrons of velocity very close to that of light; in the case of the cosmic rays the presence of electrons of energy over 10^9 volts has been established. The range of such particles, the number and nature of the secondary particles produced in their passage through matter can be computed theoretically. It has seemed to us desirable for a better understanding of the nature and properties of these radiations to have a more complete theoretical answer to these questions than is available. We shall, in this paper, be concerned with the behavior of two types of particle; chiefly we wish to study the electron of high energy; energy large, that is, compared with the proper energy mc^2. But we shall also study the impacts of a certain type of neutron, a hypothetical elementary neutral particle carrying a magnetic moment. This particle necessarily has a spin and presumably satisfies the exclusion principle; its existence was tentatively proposed by Pauli,[1] on the ground that by its introduction certain difficulties in the theory of nuclei could be resolved, and on the further ground that such a particle could be described by a wave function which satisfies all the requirements of quantum mechanics and rela-

[1] Professor Pauli presented the considerations which led him to the introduction and definition of the magnetic neutron at a seminar on theoretical physics in Ann Arbor in the summer of 1931.

tivity. These requirements, for instance, show that an elementary particle can have a magnetic but not an electric dipole moment; they do not suffice to fix the magnitude of the moment nor of the mass of the particle. Pauli supposed that such neutrons might form a third element in the building of nuclei, in addition to the electrons and protons; in this way one could understand the anomalous spin and statistics of certain nuclei, and the apparent failure of the conservation of energy in beta-particle disintegration. Pauli accordingly supposed that the mass of the neutron was not much greater than that of the electron, and that its magnetic moment was small compared to the Bohr magneton. One may, however, assume that the neutron has a mass very close to that of the proton, and that such neutrons are substituted for pairs of electrons and protons in certain nuclei, instead of being added to them; such neutrons would help explain the anomalous spin and statistics of nuclei, although they would throw no light on the beta-ray disintegrations. The experimental evidence on the penetrating beryllium radiation suggests that neutrons of nearly protonic mass do exist; and since our calculations may be carried through without specifying the mass or magnetic moment of the neutron, we shall consider the most general particle which satisfies the wave equation proposed by Pauli. It is important to observe that there may very well be other types of neutral particles, which are not elementary, and to which our calculations do not apply; and for clarity we shall call the particle which satisfies Pauli's wave equation a magnetic neutron. Certain of our results, such as the relatively great penetration of the particle, relatively rare impacts, and large mean energy loss per impact, characterize the behavior of any neutral particle.

The collisions and range of beta-particles have been often studied theoretically; and even the case in which the primary velocity of the beta-particle is very close to that of light has been studied by Bohr[2] in his classical theory of range. But Bohr used a classical model for the atom and the beta-particle, and a classical method for treating their interaction; and Bohr's treatment of the close impacts involving large energy losses is even classically not free from ambiguity. A very complete quantum theoretical calculation has been made by Bethe[3] for the case that the electron has a velocity not comparable with that of light; and our first problem in this paper is to make this calculation relativistically, so that it may be applied to electrons of energy very large compared to their proper energy.

At first sight the semi-classical method used by Gaunt[4] would appear appropriate for our purpose. Gaunt's method is semi-classical in that, in the quantum mechanical expression for the probability of excitation (ionization) of an atom, the matrix component of the interaction energy between primary and atomic electrons is in part replaced by a Fourier component, replaced, that is, by the matrix component for the transition of the atomic electron of the Fourier component of the potential of the primary electron. For transi-

[2] N. Bohr, Phil. Mag. **30**, 58 (1915).
[3] H. Bethe, Ann. d. Physik **6**, 325 (1930).
[4] J. A. Gaunt, Proc. Cam. Phil. Soc. **23**, 732 (1928).

tions in which the primary electron has a small change of momentum, small energy loss and small deflection, this method constitutes a valid application of the correspondence principle. And this method is readily extended to high velocity primaries, since we have now only to take, in place of the electrostatic potential, the Fourier component of the retarded four-vector potentials of the primary as a perturbation which induces transitions of the atomic electrons. The method is, however, not very elegant, even for the computation of the probability of small energy losses, because it gives an incorrect and very large probability of transitions involving small energy losses but large deflections, transitions to the study of which it may not legitimately be applied. We shall, therefore, not use this method; but in §II we shall give an outline of it, because it gives an illuminating insight into our formulae, and makes clear in a simple way the reason for the increase in ionization power with increasing energy of the primary. Quite recently Møller[5] has given a beautiful method of treating the relativistic impact of two electrons. This method is based upon a refinement of the correspondence principle; it neglects higher powers of the interaction energy between the electrons, and the effect of radiative forces; but within these limits it is strict and unambiguous, and enables one to take account, not only of the relativistic variation of mass with the velocity of the electrons, but of the retardation of the forces between them, of the spin forces, of interchange and the exclusion principle. The method is applicable not only to the impacts between two free electrons, but to the impacts of a free and an atomic electron in which small energies are transferred; it is further applicable to the impact of a neutron with an electron or proton; and it is this method which we shall use. In doing this we may take advantage of the fact that for energy losses very large compared to the ionization energy of the atomic electron we may treat both electrons as free; whereas for small energy losses we may neglect relativistic effects for the atomic electron, and, as it turns out, interchange. For extranuclear electrons there is a region of energy loss large compared to the binding energy and small compared to the proper energy where the two calculations merge and agree. For the neutron the probability of small energy losses is small, and the binding of an atomic electron can be largely neglected.

In §I then we shall give an outline of Møller's method, and its application to the three calculations; intimate collisions with an electron, collisions with a bound electron involving small energy loss, and collisions of a neutron. We shall have to discuss here the limitations imposed upon the method by the neglect of higher powers of the interaction energy and of radiative forces. These points cannot be fully settled without an adequate quantum electrodynamics, since they involve questions whose classical analogue is the theory of the structure of the electron. But we shall see that our method gives an upper limit to the range and a lower limit to the ionizing power; and we shall see further that the neglect we have made is small of the order of the fine structure constant. We shall have to carry the details of the calculation of the

[5] C. Møller, Zeits. f. Physik **70**, 786 (1931). Further L. Rosenfeld, ibid. **73**, 253 (1931).

impact of two free electrons rather further than was done by Møller, and shall obtain a formula for the differential probability of a given energy loss, a formula which may be applied to the impact of an electron with an atomic electron for all energy losses large compared to the binding energy.

In §II we shall treat impacts involving small energy losses, and compute the range and the number of primary ions for a beta-particle. In §III we shall carry through the calculations of the impacts of a magnetic neutron, and apply them for a few typical values of the constants characterizing the particle.

We shall give here a summary of certain of our results. Let the energy of the primary electron be

$$E = \epsilon mc^2 \tag{I}$$

and the binding energy of the second electron be I, and the energy lost by the primary be E'. We have to consider only collisions in which

$$E' \leqq \tfrac{1}{2}[(\epsilon - 1)mc^2 - I] \tag{II}$$

since one particle has always at least half the initial energy (see 1.32). The following formulae hold for large ϵ.

1.

The differential cross section for an energy loss $E' \gg I$ is

$$\sigma dE' = \frac{2\pi e^4}{mc^2} \frac{dE'}{E'^2} \cdot \frac{E^4 + E'^4 + (E - E')^4}{2E^2(E - E')^2}. \tag{III}$$

The angle γ between the trajectories of the two electrons after such an impact is given by

$$\gamma^2 = (2mc^2/E')(1 - E'/E).$$

2.

If the cross section for ionization of the atomic electron by a primary electron of velocity v be

$$\sigma_{\text{ion}} = k_1(2\pi e^4/mv^2 I) \ln mv^2/k_2 I \tag{IVa}$$

then the cross section for ionization by the fast electron of energy ϵmi^2 is

$$\sigma_{\text{ion}} = k_1(2\pi e^4/mc^2 I)\{\ln (mc^2/k_2 I) + 2 \ln \epsilon\}. \tag{IVb}$$

When the atomic electron was initially bound in the normal state of a hydrogen-like atom,

$$k_1 = 0.29; \; k_2 = 0.024. \tag{IVc}$$

3.

If the mean energy loss to the atomic electron for a primary of velocity v is

$$\delta E = (4\pi e^4/mv^2) \ln mv^2/k_3 I \tag{Va}$$

then for a primary of large energy

$$\delta E = (4\pi e^4/mc^2)\{\ln mc^2/k_3 I + (3/2) \ln \epsilon + 0.22\}. \tag{Vb}$$

For an electron in the normal state of a hydrogen-like atom

$$k_3 = 1.1. \tag{Vc}$$

4.

When a magnetic neutron hits a charged particle, the mean energy loss per impact is of the same order as the maximum energy loss permitted by the conservation laws. If the mass of the neutron be M, its magnetic moment μ, its velocity small compared to that of light and its energy E, and if the charge and mass of the secondary be respectively e, and $m=\lambda M$, then the mean energy loss of the neutron is

$$\delta E = (7\pi^3 e^2\mu^2/6h^2c^2)E \tag{VIa}$$

for $\lambda=1$

and

$$\delta E = (2\pi^3 e^2\mu^2/h^2c^2)\lambda E \tag{VIb}$$

for $\lambda \ll 1$. Further details and other cases of impacts of the magnetic neutron are treated in §III.

NOTE: After the completion of this work, a very interesting paper[6] of Heisenberg has come to hand. Heisenberg is concerned with the problem of the nature of cosmic rays; and he derives theoretical formulae for the range of high speed particles with which to compare the experimental findings. Heisenberg's formula for the range of a fast electron differs only very slightly from that which we have found (Vb); we do not believe that the application of our formulae would lead to sensibly different conclusions. Note added in proof: An outline of a derivation of a formula for the energy loss of a fast electron, by a method also based on Møller's and altogether similar to that used by us in II, has been given in a recent paper of Bethe (Zeits. f. Physik, **76,** 283 (1932)).

I. Relativistic Theory of Impacts

The method which Møller proposed for the relativistic treatment of impacts is a generalization of two familiar elementary methods; the nonrelativistic collision theory of the quantum mechanics, and the theory of the transitions induced in a quantum mechanical system by a known electromagnetic field. According to the nonrelativistic quantum mechanics, we can write down the interaction energy V of the two particles which are colliding as a function of the coordinates (and in some cases momenta) of the particles; for two electrons this energy is just the electrostatic interaction energy. We can further specify the stationary states of the non-interacting particles by certain quantum numbers r, ρ, e.g., the components of momenta of the two particles. In the matrix scheme in which these quantum numbers are diagonal there will be a matrix $V_{rs}{}^{\rho\sigma}$ which corresponds to the interaction energy. Then with neglect of higher powers of the interaction energy, and with suitable normalization, the transition probability for a transition in which the one particle changes its state from $s\rightarrow r$ and the other from $\sigma\rightarrow\rho$ will be given by

$$P_{rs}{}^{\rho\sigma} = (4\pi^2/h)\left| V_{rs}{}^{\rho\sigma}\right|^2 \delta(E_r + E_\rho - E_s - E_\sigma). \tag{1.1}$$

[6] W. Heisenberg, Ann. d. Physik [5,] **13,** 430 (1932).

(The energy of a particle in state r is written E_r; and $\delta(x)$ is the delta function.) Further, when we know the wave functions for the particles in their non-interacting stationary states

$$\begin{aligned} \psi_s &= u_s e^{(-2\pi i/h)\cdot E_s t} \\ \psi_\sigma &= u_\sigma e^{(-2\pi i/h)\cdot E_\sigma t} \end{aligned} \tag{1.2}$$

we can compute the matrix $V_{rs}{}^{\rho\sigma}$:

$$V_{rs}{}^{\rho\sigma} = \int\int d\mathbf{r}d\mathbf{r}'\tilde{u}\tilde{u}_\rho' V u_s u_\sigma'. \tag{1.3}$$

(Integral over configuration space $\mathbf{r}$, $\mathbf{r}'$ of two particles.) Now when we are dealing with particles whose velocity is close to that of light, we must of course use relativisitic wave functions to characterize their stationary states; there is no difficulty in doing this. For the electron we must use solutions of Dirac's wave equation; for the neutron we may use solutions of the wave equation given by Pauli. But the function V no longer exists, since one cannot, when the retardation of the forces is taken into account, express the interaction energy as a function of the coordinates and momenta alone. What then, in the relativistic theory, should replace the matrix element $V_{rs}{}^{\rho\sigma}$?

We can answer this question if we consider first the perturbation induced in a quantum mechanical system by a given electromagnetic field. Let the four-vector potential of the field be $\phi_\mu(x, y, z, t)$; $\mu = 1 \cdots 4$, and the charge and current density vector of the system be

$$J^\mu = \psi j^\mu \psi. \tag{1.4}$$

Then

$$V = -1/c\phi_\mu j^\mu \tag{1.5}$$

is the operator representing the interaction energy of field and system. We resolve this operator in a Fourier integral

$$V = \int V_\nu e^{-2\pi i\nu t}d\nu; \; V_\nu = (-1/cj^\mu)\int e^{2\pi i\nu t}\phi_\mu dt. \tag{1.6}$$

To V_ν there now corresponds a matrix in the scheme of the stationary states without field

$$V_\nu{}^{\rho\sigma} = \int u_\rho V_\nu \mu_\sigma d\mathbf{r}. \tag{1.7}$$

Here, again neglecting higher powers of the interaction energy, we find for the transition probability

$$4\pi^2/h \left| V_\nu{}^{\rho\sigma} \right|^2 \delta(E_\rho - E_\sigma - h\nu). \tag{1.8}$$

Now when the field ϕ_μ is produced by a particle of known trajectory, a trajectory uninfluenced by the reaction of the system upon it, we express the potentials in terms of the retarded charge and current density J'^μ of this particle:

$$\phi^{\mu}(\mathbf{r}, t) = (1/c)\int d\mathbf{r}' \frac{[J'^{\mu}]}{r} t - r/c;\ r = |\,\mathbf{r} - \mathbf{r}'\,|\,. \tag{1.9}$$

Then

$$V_{\nu} = (-\,1/c^2)\int d\mathbf{r}'\left[j_{\mu}\int J'^{\mu}e^{2\pi i\nu t}dt\right](e^2/r)^{\pi i\nu r/c} \tag{1.10}$$

and the transition probability $\sigma \rightarrow \rho$ is given by

$$(4\pi^2/h)\,|\,V_{\nu}{}^{\rho\sigma}\,|^2\,\delta(E_{\rho} - E_{\sigma} - h\nu)$$

with

$$V_{\nu}{}^{\rho\sigma} = (-\,1/c^2)\int\int d\mathbf{r}d\mathbf{r}'\tilde{u}_{\rho}\left[j_{\mu}\int J'^{\mu}e^{2\pi i\nu t}dt\right]\frac{e^{2\pi i\nu r/c}}{r}\,u_{\sigma}. \tag{1.11}$$

This at once suggests that, when the trajectory of the particle is influenced by the reaction of the system, so that the state of the particle also changes as a result of the interaction, we replace the Fourier component of the retarded potentials of the particle by the matrix component corresponding to the transition in question. Thus if again

$$J'^{\mu} = \tilde{\psi}j'^{\mu}\psi$$
$$\psi_r = u_r e^{(-2\pi i/h)E_r t}$$

are the wave functions for the stationary states of the particle, we have to replace $\int J'^{\mu}e^{2\mu i\nu t}dt$ by $\tilde{u}_r j'^{\mu}u_s$ with $E_s - E_r = h\nu$.
This gives

$$\begin{aligned} P_{rs}{}^{\rho\sigma} &= (4\pi^2/h)\,|\,V_{rs}{}^{\rho\sigma}\,|^2\,\delta(E_r + E_{\rho} - E_s - E_{\sigma}) \\ V_{rs}{}^{\rho\sigma} &= (-\,1/c^2)\int\int d\mathbf{r}d\mathbf{r}'\tilde{u}_r\tilde{u}_{\rho}'j_{\mu}j'^{\mu}e^{(2\pi i/hc)(E_s - E_r)r}u_s u_{\sigma}'/r. \end{aligned} \tag{1.12}$$

This reduces to (1.3) when retardation may be neglected and the velocity of the particles is small compared to that of light; it reduces to (1.11) when the reaction of the system on the particle may be neglected.

This is the formula proposed by Møller.[5] Although the derivation of the formula distinguishes between the system and the particle, the result which we obtain for $V_{rs}{}^{\rho\sigma}$ when we reverse the roles of the two

$$(-\,1/c^2)\int\int d\mathbf{r}d\mathbf{r}'\tilde{u}_{\rho}\tilde{u}_r'j^{\mu}j_{\mu}'e^{2\pi i/hc(E_{\sigma} - E_{\rho})r}u_s'u_{\sigma}/r.$$

gives the same transition probability, and the same matrix to represent the interaction energy.

We are thus led to consider the interaction energy represented by the matrix (1.12), $V_{rs}{}^{\rho\sigma}$. One might at first suppose that by the use of this expression for the energy, and the higher approximations of the method of variation of constants by which the transition probabilities may be computed, one could obtain a strict expression for these transition probabilities. This is not so, however, because in the derivation of (1.12) the reaction of each particle to its own field had been neglected; this reaction should be taken into account in any strict theory, and its omission makes the higher approximations —involving higher powers of the charge or moment of the particles, invalid. Thus (1.12) gives no account of collisions between the particles in which radiation is emitted; but we know that such transitions will in fact occur. Their relative frequency, frequency relative to the transitions in which no radiation is emitted, is known to be of the order

$$\alpha(\bar{v}/c)^2 \tag{1.13}$$

where α is the fine structure constant, and $\bar{v}$ the mean velocity of the charges. This probability can thus be neglected when one of the particles moves with a velocity small compared to light; even when this is not so, radiative processes are presumably relatively rare because of the smallness of α. We shall see that in our theory of the impact of a high velocity electron upon matter the impacts in which very little energy is lost are the only ones of importance. But in these impacts the secondary electron has very low velocities, so that we know that radiative processes cannot be important in these impacts. For this reason the results we obtain with neglect of radiative forces furnish a lower limit to the ionizing power and energy loss of the fast electron. On the other hand the formulae derived from (1.12) for impacts in which large energies are transferred may be seriously in error; this error is at least of the order α.

When the two colliding particles are electrons, (1.12) must be modified to take account of interchange and the exclusion principle. When the interaction energy matrix corresponds to an operator in configuration space, it is known that this may be done by writing for the interaction energy

$$\frac{1}{2} \sum_{r,s\ \rho,\sigma} V_{rs}{}^{\rho\sigma} a_\rho{}^+ a_r{}^+ a_s a_\sigma \tag{1.14}$$

and by treating the a's as dynamical variables which satisfy

$$a_\rho{}^+ a_\sigma + a_\sigma a_\rho{}^+ = \delta_{\rho\sigma}; \quad \begin{array}{l} a_\rho a_\sigma + a_\sigma a_\rho = 0 \\ a_\rho{}^+ a_\sigma{}^+ + a_\sigma{}^+ a_\rho{}^+ = 0. \end{array} \tag{1.15}$$

The order of the a's has been so chosen that the interaction of each particle with its own field has been eliminated. Now although $V_{rs}{}^{\rho\sigma}$ corresponds to no operator in configuration space, and there is no wave equation in the configuration space of the two particles with which to compare our results, we still take (1.14) to represent the interaction energy. Since each $a_r{}^+$ corresponds to a transition in which a particle enters the state r, and each a_r to one in which a particle leaves the state r, there are now four terms in this sum which give rise to a transition in which the two particles go from the states s and σ to r and ρ. Because of the symmetry of $V_{rs}{}^{\rho\sigma}$, these four terms are equal in pairs; for the transition probability we find, in place of (1.12)

$$P_{rs}{}^{\rho\sigma} = (4\pi^2/h) \left| V_{rs}{}^{\rho\sigma} - V_{r\sigma}{}^{\rho s} \right|^2 \delta(E_r + E_s - E_\rho - E_\sigma). \tag{1.16}$$

In dealing with the impacts of a fast electron on an atom, it is easy to see that when the energy transferred to the atomic electron is small the interchange terms in (1.16) are negligible; for such impacts we may use (1.12) in place of (1.16).

We have now to apply these formulae to the three cases: (a) Impacts of two free electrons. (b) Impacts of a fast electron with an atomic electron, in which little energy is given to the secondary. (c) Impacts of a neutron with an electron or proton.

(a) High velocity impacts of free electrons

To apply (1.16) to the impacts of two free electrons we have to write down explicitly the wave functions for the stationary states of the free electrons and the expressions for the matrix components of the charge and current vector. The states of each electron we may specify by giving the components of momentum

$$\boldsymbol{p}^{(1)} = (p_x^{(1)}, p_y^{(1)}, p_z^{(1)}); \ \boldsymbol{p}^{(2)} = (p_x^{(2)}, p_y^{(2)}, p_z^{(2)})$$

and the component of spin $\sigma^{(1)}$, $\sigma^{(2)}$ of each electron in the z direction. (We exclude states of negative energy; and to the approximation here considered this causes no ambiguity.) For each of the electrons we take the wave functions to be solutions of the equations

$$[(h/2\pi i)(\partial/\partial t) + (hc/2\pi i)(\alpha \text{ grad}) - \alpha_0 mc^2]\psi = 0 \tag{1.17}$$

in which for each electron we take the α's in the familiar form

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};$$
$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \quad \alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.18}$$

The solutions of these equations we write in the form

$$\psi_{p\sigma}{}^{j} = a_{p\sigma}{}^{j} e^{(2\pi i/h)[(\boldsymbol{p}\cdot\boldsymbol{r})-Et]} \tag{1.19}$$

with

$$E = c(m^2c^2 + p^2)^{1/2}$$

and with the $a_{p\sigma}{}^{j}$ functions of $(\boldsymbol{p}, \sigma)$. For an electron initially at rest we take for the a^j:

$$\begin{array}{ll} \sigma = +1 & \sigma = -1 \\ a_1 = a_2 = a_4 = 0;\ a_3 = 1; & a_1 = a_2 = a_3 = 0;\ a_4 = 1. \end{array} \tag{1.20}$$

These states will be characterized by an index (0). Thus

$$\boldsymbol{p}^{(0)} = 0,\ E^{(0)} = mc^2;\ \sigma^{(0)} = \pm 1;\ \text{etc.}$$

This wave function represents a uniform electron density of one electron per unit volume. For an electron moving in the z direction we take

$$\begin{array}{ll} \sigma = +1 & \sigma = -1 \\ a_2 = a_4 = 0 & a_1 = a_3 = 0 \\ a_1 = -Np_z/g & a_2 = Np_z/g \\ a_3 = N & a_4 = N \end{array} \tag{1.21}$$

where

$$g = mc + E/c$$

and

$$N^2 = 2g\,(E^2 - m^2c^4)^{1/2}.$$

These states will be characterized by a superscript (i). This wave function is normalized to represent a stream of unit flux, so that the transition probabilities computed from it will give directly the cross section for scattering.

We may without loss of generality suppose that after impact both electrons are moving in the xz plane, since, as we shall see, momentum is conserved in the impact. For such an electron we take

$$\begin{array}{ll} \sigma = +1 & \sigma = -1 \\ a_1 = -Np_z/g & a_1 = -Np_x/g \\ a_2 = -Np_x/g & a_2 = Np_z/g \\ a_3 = N & a_3 = 0 \\ a_4 = 0 & a_4 = N \end{array} \tag{1.22}$$

where

$$g = mc + E/c$$

and

$$N^2 = -\,gc/2Eh^3.$$

These states will be characterized by superscripts (1) and (2). This wave function is normalized to $dp_x\,dp_y\,dp_z = d\boldsymbol{p}$. We shall need the Jacobian

$$J^{(1)} = \partial(p_x^{(1)}, p_y^{(1)}, p_z^{(1)})/\partial(E^{(1)} + E^{(2)}, \vartheta^{(1)}, \phi^{(1)})$$

where, by the conservation of momentum,

$$E^{(2)}(p^{(2)}) = E^{(2)}(p^{(i)} - p^{(1)}). \tag{1.23}$$

We find

$$j^{(1)} = \frac{2mp^{(1)}E^{(1)}E^{(2)}\sin\vartheta^{(1)}}{mc^2(1 + \cos^2\vartheta^{(1)}) + E\sin^2\vartheta^{(1)}}$$

Note: There is at this point in Møller's paper an error, in that Møller uses in place of j^1

$$j^{(1)} = \partial(p_x^{(1)}, p_y^{(1)}, p_z^{(1)})/\sigma(E^{(1)}, \vartheta^{(1)}, \phi^{(1)}).$$

Now $J^1 \neq J^{(1)}$, because by the conservation laws $E^{(2)}$ depends on $p^{(1)}$. It is $J^{(1)}$ that we must use to take out the $\delta(E^{(0)}+E^{(i)})-(E^{(1)}+E^{(2)})$ in (1.16). We are indebted to Dr. Heisenberg for tellung us that Møller had found an error at this point in his paper.

Further the charge density is given by

$$e\sum_{j=1}^{4}\tilde{\psi}^j\psi^j \tag{1.24}$$

and the current density by

$$ec\sum_{j,k}\tilde{\psi}^j\alpha jk\psi^k. \tag{1.25}$$

If we put these expressions in (1.16), we find for the transition probability from the intial states [(1.20) and (1.21)] of the two electrons to states

$$\boldsymbol{p}^{(1)},\ \sigma^{(1)},\ \boldsymbol{p}^{(2)}\sigma^{(2)}$$

such that

$$E^{(1)} + E^{(2)} = E^{(0)} + E^{(i)}$$

the expression

$$4h^3e^4J^{(1)}|A|^2\left|\int d\boldsymbol{r}e^{2\pi i/h([\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(1)}-\boldsymbol{p}^{(2)}]\cdot\boldsymbol{r})}\right|^2 d\boldsymbol{p}^{(2)}d\vartheta^{(1)}d\phi^{(1)} \tag{1.26}$$

$$A = \frac{A_{1,i}{}^0A_{2,0}{}^0 - (\boldsymbol{A}_{1,i}\cdot\boldsymbol{A}_{2,0})}{|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(1)}|^2 - (1/c^2)(E^{(i)}-E^{(1)}|^2} - \frac{A_{1,0}{}^0A_{2,i}{}^0 - (\boldsymbol{A}_{1,0}\cdot\boldsymbol{A}_{2,i})}{|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(2)}|^2 - (1/c^2)(E^{(i)}-E^{(2)})^2}. \tag{1.27}$$

Here

$$A_{1,0}^{0} = \sum_{j=1}^{4}\tilde{a}^j(p^{(1)},\sigma^{(1)})a^j(p^{(0)},\sigma^{(0)}).$$

$$\boldsymbol{A}_{1,0} = \sum_{j,k=1}^{4}\tilde{a}^j(p^{(1)},\sigma^{(1)})\alpha jka^k(p^{(0)},\sigma^{(0)}).$$

Now

$$Q = \lim_{V\to\infty}\left|\int_V d\boldsymbol{r}e^{(2\pi i/h)([\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(1)}-\boldsymbol{p}^{(2)}]\cdot\boldsymbol{r})}\right|^2 \to V\delta(\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(1)}-\boldsymbol{p}^{(2)}). \tag{1.28}$$

Thus momentum as well as energy are conserved, and we may set

$$\boldsymbol{p}^{(2)} = \boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}. \tag{1.29}$$

Further if we divide (1.27) by V we get the cross section for impact with a single electron at rest. In (1.27) we have to consider all the possible orientations of spin of the two electrons in initial and final states: to take one fourth the sum of (1.27) over the sixteen combinations of values of $\sigma^{(i)}$, $\sigma^{(0)}$, $\sigma^{(1)}$, $\sigma^{(2)}$. The resulting cross section can be expressed as a function of ϵ and ϑ, the angle of deflection of one of the electrons; it is independent of the azimuth of deflection of this electron; and all the components of momentum are determined by the conservation laws when ϑ and ϕ are given for one electron. Before carrying out this reduction we may make a few observations on the geometry of the impact.

According to the conservation laws, if the two electrons after impact have direction of motion making the angles ϑ_1, ϑ_2 with the z axis,

$$p^{(1)} \sin \vartheta_1 = p^{(2)} \sin \vartheta_2. \tag{1.30}$$

Further, the energy of a particle coming off at an angle ϑ is

$$E' = mc^2 \frac{2\epsilon - (\epsilon - 1)\beta^2}{2 + (\epsilon - 1)\beta^2}; \quad \begin{array}{l} \alpha = \cos \vartheta; \\ \beta = \sin \vartheta; \end{array} \tag{1.31}$$

and the absolute value of its momentum is $mc[2\alpha(\epsilon^2-1)^{1/2}]/[2+(\epsilon-1)\beta^2]$. Thus the particle with the smaller energy always comes off at a larger angle; when the energy is shared equally the two particles come off at the same angle ϑ_m such that

$$\sin^2 \vartheta_m = 2/(\epsilon + 3). \tag{1.32}$$

In general the angle γ between the particles is

$$tg\gamma = [2 + (\epsilon - 1)\beta^2]/[(\epsilon - 1)\alpha\beta] \tag{1.33}$$

which reduces to

$$\gamma^2 = (2mc^2/E')(1 - E'/E) \tag{1.34}$$

for large ϵ. [Here E' is the smaller of the two energies $E^{(1)}$, $E^{(2)}$.] In no impact do both electrons come off at an angle larger than ϑ_m; and since (0. 7) gives the probability that an electron come off at an angle ϑ, and the other at the corresponding angle determined by (1.30), we have only to consider impacts in which $\vartheta \leqq \vartheta_m$; impacts in which $\vartheta > \vartheta_m$ will be the same impacts as those for which $\vartheta < \vartheta_m$, but in which the angle ϑ refers to the slower and not the faster of the two electrons.

We should note that by (1.34) the angle between the electrons grows smaller as the both the primary energy and the energy transferred grow large compared to the proper energy. This point is of some importance in connection with the interpretation of cloud chamber experiments, since it leads us to expect that very high energy secondary electrons produced by elastic impact with high energy primaries will come off nearly parallel to the direction of the primary.

We shall let then ϑ be the angle which the faster of the two secondary electrons, for which we use (1), makes with the primary direction. Further we shall write $\alpha = \cos\vartheta$, $\beta = \sin\vartheta$, $q = (\epsilon - 1)\beta^2$. Then the conservation laws give us the relations

$$\begin{aligned}
p_x^{(1)} &= -p_x^{(2)} = [(\epsilon^2-1)^{1/2}2\alpha\beta/(2+q)]mc \\
p_z^{(i)} &= (\epsilon^2-1)^{1/2}mc \\
p_z^{(1)} &= [2\alpha^2(\epsilon^2-1)^{1/2}/(2+q)]mc \\
p_z^{(2)} &= [(\epsilon^2-1)^{1/2}(\epsilon+1)\beta^2/(2+q)]mc \\
|\mathbf{p}^{(i)}-\mathbf{p}^{(1)}|^2 - (1/c^2)(E^{(i)}-E^{(2)})^2 &= [2(\epsilon^2-1)\beta^2/(2+q)]m^2c^2 \\
|\mathbf{p}^{(i)}-\mathbf{p}^{(2)}|^2 - (1/c^2)(E^{(i)}-E^{(2)})^2 &= [4(\epsilon-1)\alpha^2/(2+q)]m^2c^2
\end{aligned}$$

$$\begin{aligned}
g^{(0)} &= 2mc & g^{(2)} &= [(4+(\epsilon+3)q)/(2+q)]mc \\
g^{(i)} &= (\epsilon+1)mc & p^{(1)} &= [2\alpha(\epsilon^2-1)^{1/2}/(2+q)]mc \\
g^{(1)} &= [2(\epsilon+1)/(2+q)]mc & E^{(1)} &= [(2\epsilon-q)/(2+q)]mc^2.
\end{aligned} \tag{1.35}$$

Thus we get for the differential cross section,

$$\sigma d\alpha = \frac{\pi e^4}{m^2c^4}\cdot\frac{(\epsilon+1)^2\alpha}{2(\epsilon-1)^2}\cdot\frac{4+(\epsilon+3)q}{(2+q)^2}\cdot d\alpha \sum_{\sigma^{(0)},\sigma^{(1)},\sigma^{i},\sigma^{(2)}}^{\pm} \left|\frac{L(\sigma)}{(\epsilon+1)\beta^2} - \frac{M(\sigma)}{2\alpha^2}\right|^2 \tag{1.36}$$

Here each of the $L(\sigma)$, $M(\sigma)$ refers to a different one of the sixteen possible orientations of electron spin; $L(\sigma)$ is the direct term, $M(\sigma)$ the interchange term. The L and M can be tabulated as a function of ϵ and ϑ. We write

$$y = (\epsilon-1)/(\epsilon+1);\ S^{-1} = 4 + (\epsilon-1)(\epsilon+3)\beta^2;$$

$\sigma^{(0)}$	$\sigma^{(i)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	$L(\sigma)$	$M(\sigma)$
+ −	+ −	+ −	+ −	$1+y\alpha^2-(1+\alpha^2)(\epsilon^2-1)\beta^2S$	$1-y\alpha^2+(\epsilon^2-1)\beta^4S$
− +	+ −	− +	+ −	$1+y\alpha^2+(\epsilon-1)[4\alpha^2\beta^2$ $-(1+\alpha^2)(\epsilon+1)\beta^2]S$	$-2y\alpha^2[1+3\alpha^2+\epsilon\beta^2]S$
− +	+ −	+ −	+ −	$\mp 2\alpha\beta(\epsilon-1)(2+q)S$	$\pm 2\alpha\beta y[2(\epsilon+2)\alpha^2+(\epsilon+1)^2\beta^2)S$
+ −	+ −	+ −	− +	$\pm 2\alpha\beta(\epsilon-1)(1+\alpha^2)S$	$\mp 2\alpha\beta qS$
− +	− +	+ −	− +	$\mp 4\alpha^3\beta yS$	$\pm 2\alpha\beta y[2\alpha^2+(\epsilon+1)\beta^2]S$
+ −	− +	+ −	+ −	$\mp 4y\alpha\beta(1+\epsilon\beta^2)S$	$\pm 2y\alpha\beta[4\alpha^2+(\epsilon+1^2)\beta^2]S$
+ −	+ −	− +	− +	$2\alpha^2qS$	$2\alpha^2qS$
− +	+ −	− +	+ −	$-2q(1+\epsilon\beta^2)S$	$1-y\alpha^2+q[4\alpha^2+(\epsilon+1)\beta^2]S$

(1.37)

For large ϵ the last six contribute nothing to the cross section; the first four are important for energy losses not large compared to the proper energy; the next four are important for all energy losses comparable with the proper energy; the next two are important only for energy losses comparable with the primary energy. From (1.35) the energy transferred is

$$E' = mc^2(\epsilon + 1)q/2 + q. \tag{1.38}$$

We must consider only impacts for which

$$\theta \leqq \theta_m, \; E' \leqq \tfrac{1}{2}(\epsilon - 1)\cdot mc^2. \tag{1.39}$$

For these the differential cross section is given by

$$\sigma dE' = \frac{2\pi e^4}{mc^2}\frac{dE'}{E'^2}\cdot\frac{E^4 + E'^4 + (E - E')^4}{2E^2(E - E')^2}. \tag{1.40}$$

This concludes our study of the impacts in which large energies are transferred. Certain further calculation by this method, primarily for the case of smaller ϵ, were given[5] by Møller; and there, too, the connection between certain results of this calculation with earlier attempts at the calculation of impacts were given. One point we may mention here: when ϵ is not large, we may use Rutherford's formula to give us the differential cross section for energy loss,—or rather the nonrelativistic quantum mechanically extension of Rutherford's formula which takes account of interchange and the exclusion principle. This gives, for $E' \ll E$,

$$\sigma dE' \cong 2\pi e^4 dE'/EE'^2. \tag{1.41}$$

This agrees with our result for energy losses small compared to the proper energy; but for larger energy transfers it gives too large a cross section. When higher powers of the interaction energy are not neglected, terms of the order of the fine structure constant must be added to (1.40). In the case of the nonrelativistic calculation (1.41) these are only of importance when E' is of the same order as E; and presumably, although not certainly, these terms will not be of importance for large ϵ except in this case $E' \sim E$.

(b) Small energy transfers to bound electrons

We have now to consider the case that a primary electron of very large energy gives to a secondary which was originally bound in an atom an energy not comparable to its proper energy. The states for the primary electron are still given by (1.21) and (1.22); but certain simplifications are introduced even here by the fact that the energy of the primary changes very little. For practically all such impacts involve a very small deflection for the primary; and in their treatment it is legitimate to neglect quantities which are small when the primary momentum changes by a relatively very small amount. With this understanding we see that only the matrix components of the charge and current of the primary in which the spin of the primary does not change are important, and that these are independent of the original orientation of the primary spin, which we may thus take parallel to z. (Case $\sigma = +1$). Further the components of current, and thus of vector potential, in the xy plane, are negligible. We have to consider only the scalar potential and the z component of the vector potential of the primary.

There are two further simplifications. In the first place interchange is negligible (interchange terms are important in general only when the energy transferred in of the order of the primary energy); we may thus use (1.12) in place of (1.16). In the second place the secondary electron never gets a velocity comparable with that of light, so that for it we may use nonrelativistic wave functions and expressions for charge and current density; and we may neglect the spin of this electron. On the other hand we may not use wave functions which neglect the binding of the atomic electron, and must replace for it (1.19) and (1.21). Let the initial energy and wave function of the electron be

$$E^{(0)} = -I; \ \psi_0 = e^{(2\pi i/h)It}u_0. \tag{1.42}$$

After the impact the atomic electron may be excited or ionized. Let the corresponding energies and wave functions be

$$\begin{aligned} E_{nl}; \ \psi_{nl} &= e^{(-2\pi i/h)E_{nl}t}u_{nl}; \\ E, \ \psi_{El} &= e^{(-2\pi i/h)Et}u_{El}; \end{aligned} \tag{1.43}$$

where l stands for the two other quantum numbers in addition to the energy, necessary in the most general case to specify the state of the electron. Further let the continuous wave functions for energies above the ionizing potential be normalized in the energy scale.

The expressions for the charge and current density are then simply

$$\begin{aligned} \rho &= e\tilde{\psi}\psi \\ J &= (e/2m)[(h/2\pi i)(\tilde{\psi}\,\mathrm{grad}\,\psi - \psi\,\mathrm{grad}\,\tilde{\psi}) \\ &\quad - (2e/c)\tilde{\psi}A\psi] \sim (eh/2\pi im)\tilde{\psi}\,\mathrm{grad}\,\psi. \end{aligned} \tag{1.44}$$

The terms in the vector potential A are of higher order in the interaction energy and may be dropped.

If as before we let the primary electron be deflected in the plane, and call the angles of deflection again ϑ, ϕ, we get from (1.12) for the probability of excitation to a state n, l:

$$\sigma_{nl} = \frac{4\pi^2 e^4}{h^4 c}\int\int d\vartheta d\phi J^{(1)} \left| V_{p^{(1)}nl} \right|^2 \qquad r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$$

$$\begin{aligned} V_{p^{(1)}nl} = \int\int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \tilde{u}_{nl}(r_2) \exp\{2\pi i/h[([\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}]\cdot\boldsymbol{r}_1) \\ - (1/c)(E^{(i)} - E^{(1)})r\}[1 - h/2\pi imc(\partial/\partial z_2)/r]u_0(r_2) \end{aligned} \tag{1.45}$$

and for the probability of ionization to a state of energy in the range dE

$$\sigma_E dE = \frac{4\pi^2 e^4}{h^4 c} dE \int\int d\vartheta d\phi J^{(1)} \sum_e \left| V_{p^{(1)}El} \right|^2.$$

To get the total probability of excitation or ionization these expressions must be summed over n, l; to get the energy loss per impact they must be multiplied by

$$E' = E_{nl} + I, \quad E' = E + I \tag{1.47}$$

and summed over l, summed and integrated over n and E. In the next section we shall evaluate these expressions as far as possible when we leave the atomic wave functions arbitrary; and we shall evaluate them in detail for the case that the wave functions are those of a hydrogen like atom. From them we shall find, for the differential cross section for an energy loss E' large compared to the ionizing energy I,

$$\sigma dE' = 2\pi e^4 dE'/mc^2E'^2$$

in agreement with (1.40); but for smaller energy losses we shall obtain results which differ very markedly from those given by (1.40).

(c) Theory of magnetic neutron

For the impacts of a neutron with a free electron we have again to use (1.12). Here the wave function for the electron before impact is given by (1.29); that after impact by (1.22); and the charge and current density of the electron by (1.24–5). We have only to find the matrix components of the charge and current density of the neutron.

The wave equation proposed by Pauli[1] for the magnetic neutron is simply related to the Dirac equation for the electron. In the absence of a field this latter may be written

$$[\gamma^\mu p_\mu - imc]\phi = 0;\ p_4 = (-\ h/2\pi i)(\partial/\partial t);\ p_0 = (h/2\pi i)(\partial/\partial x_l);\quad l = 1, 2, 3 \tag{1.48}$$

where the γ^μ satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}. \tag{1.49}$$

In the absence of a field this equation is to hold for the magnetic neutron, except that the neutron's mass M must be substituted for that of the electron. In the presence of a field new terms are to be added to the wave equation:

$$[\gamma^\mu p_\mu - iMc + P]\phi = 0;\ P = k\sigma^{\mu\nu}F_{\mu\nu}. \tag{1.50}$$

Here $F_{\mu v}$ is the field tensor,

$$\sigma^{\mu\nu} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$$

and κ is a constant which is related to the magnetic moment μ of the neutron:

$$\kappa = \mu/4c. \tag{1.51}$$

These terms are not altogether arbitrary. Thus $\sigma^{\mu\nu}$ must be an antisymmetric tensor; it may not involve the coordinates nor powers of the momenta higher than the first. Thus (1.50) turns out to be the only hermitian not identically vanishing possibility which involves the field strengths linearly. The fact that κ is real is required by the hermiticity of the term; otherwise one could not interpret the wave function of a particle which was conserved. The sign of the term is arbitrary, and determines whether the magnetic moment of the neutron is parallel or antiparallel to its spin.

From (1.50) one may deduce two conservations laws. One asserts that the divergence of the four vector

$$s^\mu = \phi^+\gamma^\mu\phi \tag{1.52}$$

vanishes, and gives the conservation law for the neutron density. The other

$$\partial J^\mu/\partial x^\mu = 0; J^\mu = (-\mu/2)(\partial/\partial)x^\nu(\phi^+\sigma^{\mu\nu}\phi) \quad (1.53)$$

gives, as we shall show, the conservation of the charge and current vector. The two four vectors-particle density and flux, charge and current density, do not, as in the case of the electron, differ merely by a constant factor.

The first justification for calling μ the magnetic moment of the neutron we obtain if we reduce (1.50) for the case of velocities small compared to that of light. We find then that the neutron is described by a two-component wave function which satisfies

$$[(h/2\pi i)(\partial/\partial t) - h^2/8\pi^2 M\Delta + \mu(\boldsymbol{H}\cdot\sigma)]\phi = 0. \quad (1.54)$$

Here the σ's are the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.55)$$

and $\boldsymbol{H}$ the magnetic field. This is just what we should expect for the wave equation of a neutral particle carrying a spin and a magnetic moment μ.

A further justification for calling μ the magnetic moment of the neutron we can obtain by showing that

$$J^\mu = (-\mu/2)(\partial/\partial x^\nu)(\phi^+\sigma^{\mu\nu}\phi) \quad (1.56)$$

does correspond to the charge and current vector. For if we calculate the potentials and field strengths of a wave packet moving with low velocity by using (1.53) for the charge and current vector, we get just the field we should expect for the neutron of magnetic moment. Here as in the case of the electron the components of momentum do not completely specify the state; we need also to specify the orientation of the spin; and by doing this we determine the orientation of the field-producing magnetic moment.

We can verify that (1.53) is right in the following way. The wave equation (1.50) and its adjoint equation may be obtained by variation of ϕ^+ and ϕ in the Lagrangian.

$$L = \int dV\phi^+[\gamma^\mu p_\mu - iMc + \kappa\sigma^{\mu\nu}F_{\mu\nu}]\phi. \quad (1.57)$$

If we add to (1.54) the Lagrangian of the empty electromagnetic field,

$$(1/16\pi)\int dVF^{\mu\nu}F_{\mu\nu} \quad (1.58)$$

and in this vary, precisely as in the familiar case of the electron, the potentials, we get in place of Maxwell's equations for the empty field

$$\partial F^{\mu\nu}/\partial x^\nu = 4\pi J^\mu \quad (1.59)$$

with J^μ given by (1.53). Thus, as far as the field produced by the neutron is concerned, J^μ given by (1.53) acts like the charge and current density. It is a four vector, and its divergence vanishes.

It is convenient to rewrite the Eqs. (1.50) in the form corresponding to (1.17). Thus for the free neutron (no field), we get

$$[(h/2\pi i)(\partial/\partial t) + (hc/2\pi i)(\boldsymbol{\alpha}\cdot\text{grad}) - \alpha_0 Mc^2]\psi = 0 \tag{1.60}$$

where the α's are given by (1.18), if we take

$$\gamma^4 = \alpha_0;\ \gamma^l = i\alpha_0\alpha_l;\ \phi = i\alpha_0\psi. \tag{1.61}$$

The density current four vector then becomes

$$\tilde{\psi}\psi,\ c\tilde{\psi}\alpha\psi. \tag{1.62}$$

The neutron density is, as it must be, essentially non-negative. The terms corresponding to P in (1.50) then become

$$-\ i\alpha_0\kappa\sigma^{\mu\nu}F_{\mu\nu} \tag{1.63}$$

and the charge and current vector is given by

$$J^\mu = -\ \mu\partial/\partial x^\nu(\tilde{\psi}\tau^{\mu\nu}\psi). \tag{1.64}$$

Here the four row matrices $\psi^{\mu\nu}$ are given in terms of the Pauli spin matrices (1.55) by

$$\begin{aligned}
&\tau^{\mu\mu} = 0\\
&\tau^{l4} = -\ \tau^{4l} = \begin{pmatrix} 0 & is_l \\ -\ is_l & 0 \end{pmatrix}\\
&\tau^{kl} = -\ \tau^{lk} = \begin{pmatrix} -\ s_m & 0 \\ 0 & s_m \end{pmatrix}.
\end{aligned} \tag{1.65}$$

Here (k, l, m) are a cyclic permutation of (1, 2, 3). This expression must be used in (1.12) to compute cross sections for impacts. For the states of the free neutron we may again use the wave functions (1.20) and (1.21). In place of (1.27) we find

$$16\pi^2 he^2\mu^2 J^{(1)} \left| B \right|^2 Q d\boldsymbol{p}^{(2)} a\vartheta^{(1)} d\phi^{(1)} \tag{1.66}$$

where Q is given by (1.28) and where now

$$B = \frac{A_{2,0}{}^0 T_{1,i}{}^4 + (\boldsymbol{A}_{2,0}\boldsymbol{T}_{1,i})}{\left|\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}\right|^2 - 1/c^2(E^{(i)} - E^{(0)})^2}\cdot \tag{1.67}$$

Here the A's are given by (1.27) and the T's are defined

$$\begin{aligned}
T_{1,i}{}^\mu &= \sum_{j,k} a^r(p^{(1)},\ \sigma^{(1)})T_{jk}{}^{\mu\nu}a^k(p^{(1)}\sigma^{(1)})\Delta p_{1,i}{}^\nu\\
\Delta\boldsymbol{p}_{1,i} &= \boldsymbol{p}^{(1)} - \boldsymbol{p}^{(1)};\\
\Delta p_{1,i}{}^4 &= -\ (1/c)(E^{(i)} - E^{(1)}).
\end{aligned} \tag{1.68}$$

As before $Q \rightarrow V\delta(\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)} - \boldsymbol{p}^{(2)})$.

We can write the cross section for a particular set of spin orientations

$$16\pi^2 he^2\mu^2 J^{(1)} \left| B(\sigma) \right|^2 d\vartheta^{(1)} d\phi^{(1)}. \tag{1.69}$$

As before we must take, to find the differential cross section, one fourth the sum of these expressions over all sixteen orientations of initial and final spins:

$$\sigma d\vartheta d\phi = 4\pi^2 h e^2 \mu^2 J^{(1)} \sum_{\sigma^{(i)},\sigma^{(1)}\sigma^{(0)},\sigma^{(2)}}^{\pm} \left| B(\sigma) \right|^2 d\delta d\phi. \qquad (1.70)$$

The quantities T occurring here may be tabulated in terms of the initial and and final momenta and energy of the neutron. If for brevity we write

$$\begin{aligned} p_z^{(i)} &= r; \quad p_z^{(1)} = r'; \quad p_x^{(1)} = p'; \\ g^{(i)} &= g; \quad g^{(1)} = g'; \qquad \Delta r = r - r'; \; \Delta g = g - g'; \end{aligned} \qquad (1.71)$$

then we find for the T's the following values:

$\sigma^{(i)}$	$\sigma^{(1)}$	T_x	T_y
$+$ $-$	$+$ $-$	$i\Delta r(rp'/gg')+i\Delta gp'/g'$	$\mp p'(1-rr'/gg')\pm\Delta rp'/gg'\pm\Delta gp'/g$
$+$ $-$	$-$ $+$	$\pm\Delta r(rp'/gg'\pm i\Delta g((r/g)+r'/g')$	$(\pm rp'^2/gg')+\Delta r(1+rr'/gg')$ $+\Delta g((r/g)+r'/g')$
$\sigma^{(i)}$	$\sigma^{(1)}$	T_z	T^4
$+$ $-$	$+$ $-$	$(-ip'^2r/gg')-i\Delta g((r/g)-r'/g')$	$(-ip'^2/g')+i\Delta r((r/g)-r'/g')$
$+$ $-$	$-$ $+$	$\pm ip'(1+rr'/gg')\pm i\Delta gp'/g'$	$\pm ip'(r/g)+r'/g')\mp i\Delta rp'/g'$

(1.72)

It is not possible, without knowing the mass of the neutron, to reduce these expressions further. We shall, therefore, postpone until §III the discussion of these formulae. There, too, we shall make such elementary investigations of the effect of the binding of the electron as are necessary to our purpose.

Before proceeding to the further application of this method of Møller, we wish again to emphasize the approximate character of the method. We have throughout neglected two things: the reaction of the particles to their own field, and all radiative processes; and higher order terms in the interaction of the two particles. In the case of the electron, both these neglects mean the omission of terms of the order of the fine structure constant, terms which are genuinely small unless both particles involved are moving with velocities comparable to light. Just what effect these terms could have on the things in which we are most interested—range and ionizing power of the primary, energy distribution of the secondaries—we shall discuss when we have our results before us. But the formula (1.40), which is derived with neglect of these terms, is subject to grave doubt, since we have no assurance that terms of the form $\epsilon\alpha$ will not appear, and no physical assurance that in the intimate collision of two electrons this approximate method can be legitimately employed. It seems at present by far the soundest method available, and perhaps experiment can show in what measure it is inadequate. The further study of impacts involving neutrons, a study which should not neglect higher powers of the interaction energy, presents even in the nonrelativistic case serious analytic complications; and we have not thought it advisable to attempt these without some more certain information about the characteristic constants, in particular the moment, of this hypothetical particle.

II. Ionizing Power and Range of Fast Electrons

We must now consider in detail the small energy losses suffered by a fast electron in its impacts with an atom. Just as in the nonrelativistic theory, if we neglect the binding of the atomic electron we find an infinite probability for small energy losses; and just as in the nonrelativistic treatment the proper consideration of the binding gives us a finite probability of energy loss and a finite range.

Before using our formula (1.46) to study these impacts we may outline briefly the semiclassical method of treating this same problem. This method, it will be remembered, is valid only for such impacts as involve small energy loss and small deflection of the primary; in this method we treat the primary particle classically, neglect the reaction of the atom upon it, and use the field calculated for this undeflected trajectory as a perturbation causing transition of the atomic electron. If we choose our coordinates so that the particle passes along the z-axis, and goes through the origin at $t=0$, then the field of the particle is given by the potentials

$$\begin{aligned} A_x = A_y = 0;\ A_z &= (ev\epsilon/c)[x^2 + y^2 + \epsilon^2(z - vt)^2]^{-1/2}; \\ \phi &= e\epsilon[x^2 + y^2 + \epsilon^2(z - vt)^2]^{-1/2}; \end{aligned} \tag{2.1}$$

where v is the velocity of the particle and

$$1/\epsilon = (1 - v^2/c^2)^{1/2}.$$

The probability of a transition of the atomic electron in which its energy changes by E' is determined by the Fourier component of this field of frequency $\nu' = E'/h$. We therefore analyze the potentials by a Fourier integral:

$$\begin{aligned} A_z &= (4e/c)\int_0^\infty K_0(2\pi\nu p/\epsilon v)\cos 2\pi\nu(t - z/v)d\nu \\ \phi &= (4e/v)\int_0^\infty K_0(2\pi\nu p/\epsilon v)\cos 2\pi\nu(t - z/v)d\nu \end{aligned} \tag{2.2}$$

where K_0 is the Hankel function:

$$K_0(\xi) = \int_0^\infty \frac{\cos\lambda d\lambda}{(\lambda^2 + \xi^2)^{1/2}};\ \text{and}\ p = (x^2 + y^2)^{1/2}.$$

The perturbation energy for the atomic electron is given by

$$V = e\phi - (1/c)A_z j_z \tag{2.3}$$

where j is the current operator which, to a sufficient approximation, is given by (1.44): $j_z = (he/2\pi mi)\partial/\partial z$. Thus the transition probability for an atomic transition $n \rightarrow n'$ in which the energy of the atom changes by $E' = E_{n'} - E_n = h\nu'$ is just

$$P_{nn'} = (4\pi^2/h^2) \left| \int \bar{u}_{n'}(2e^2/v) K_0(2\pi\nu' p/\epsilon v) e^{2\pi i \nu' z/v} [1 - (h\nu/2\pi i m c^2)\partial/\partial z] u_n d\mathbf{r} \right|^2 \quad (2.4)$$

where the u's are the wave amplitudes for the atomic electron. The nonrelativistic expression is[5]

$$P_{nn'} = \frac{4\pi^2}{h^2} \left| \int \bar{u}_{n'} \frac{2e^2}{v} K_0\left(\frac{2\pi\nu' p}{v}\right) e^{2\pi i \nu' z/v} u_n d\mathbf{r} \right|^2. \quad (2.5)$$

Now for distant collisions, to which alone this method may be legitimately applied, we may expand the potentials about the position of the nucleus of the atom, which we take to be on the y axis at a distance from the track, and consider only the first two terms of the expansion, i.e., the dipole moment of the atom. If we let the y-component of the displacement of the electron from the center of the atom be ζ then we find

$$P_{nn'}(p) = (16\pi^2 e^4/h^2 v^2) \mid \zeta_{nn'} \mid ^2 [K'(2\pi\nu' p/v\epsilon)]^2 \quad (2.6)$$

where $K'(\xi) = \partial K_0(\xi)/\partial\xi$ and $\zeta_{nn'} = \int \bar{u}_{n'} \zeta u_n d\mathbf{r}$. The nonrelativistic formula is[5]

$$P_{nn'}(p) = (16\pi^2 e^4/h^2 v^2) \mid \zeta_{nn'} \mid ^2 [K'(2\pi\nu' p/v)]^2. \quad (2.7)$$

Thus in this calculation the effect of retardation is to introduce p/ϵ in place of p in the argument of K'. Since K' is a rapidly decreasing function

$$\begin{aligned} K'(\xi) &\sim 1/\xi \text{ for } \xi \rightarrow 0, \\ K'(\xi) &\sim -(\pi/2\xi)^{1/2} e^{-\xi} \text{ for } (\xi) \rightarrow \infty, \end{aligned} \quad (2.8)$$

this means that for a given transition and a given position of the atom the probability of a given energy loss is increased by considering retardation; energy losses occur at greater and greater distances from the track as the velocity of the primary approaches more and more closely that of light. We are, therefore, led to expect a greater number of low velocity secondaries than we should find from an extrapolation of the nonrelativistic formula; and this is just what we shall find. It is an immediate consequence of the flattening out of the field of the electron in the equatorial plane.

As the position of the atom approaches the track, $p \rightarrow 0$, and the total number of transitions becomes infinite. For these intimate collisions the expansion of the potentials about p and the assumption that the track is undeflected are both illegitimate. There are a number of ways in which, for small energy losses, we might try to modify the calculation of these close impacts, in such a way that the integral over p converges and gives a finite result. Perhaps the simplest is to stop the integration at a point $\bar{p}$ where, according to (2.6), the probability of inelastic impact is unity. This point turns out to be, for a hydrogen-like electron, of the order of the Compton wave-length divided by the effective nuclear charge z:

$$\bar{p} = 1.4h/2\pi mcz \sim h/2\pi mcz. \quad (2.9)$$

This procedure gives for the mean energy loss due to small energy transfer, small compared to the proper energy,

$$\delta E = (4\pi e^4/mc^2) \ln (mc^2\epsilon/W) \tag{2.10}$$

where W is of the order of the ionizing potential I, and for a hydrogen like atom is $W=3.0\ I$. This result is in remarkable agreement with that which we shall obtain by our strict calculations with the help of (1.46); and it is of some interest to inquire into the ground for this. The answer is very simple. We shall see that in our strict calculations (2.6) is right for distant impacts, impacts, that is, in which the deflection of the primary is negligible. For angles of deflection ϑ which are much smaller than

$$\bar{\vartheta} = (2mI)^{1/2}/mc\epsilon \tag{2.11}$$

we get just this result (2.6), if we connect ϑ and the parameter p by the dynamical relation $p\vartheta \sim e^2/mc^2\epsilon$. For larger ϑ i.e., smaller p, the transition probability for transitions involving small energy loss decreases rapidly, instead of becoming infinite as it does by (2.6). Thus it is approximately right to break off the integral over p at a point

$$\bar{p} \sim (e^2/mc^2\epsilon\bar{\vartheta}) \sim (h/2\pi mcz)$$

and this is what, in these preliminary calculations, we have done.

We must now turn to the strict calculation. We shall here be dealing only with small energy losses, and since our expressions converge very rapidly as the angle of deflection ϑ increases, we have only to evaluate them for small ϑ. As before, we shall assume throughout that ϵ is a large number, and neglect higher powers of $1/\epsilon$. We need to know the change of momentum of the primary in terms of the energy lost E' and the angle of deflection ϑ. Thus, with these approximations,

$$\begin{aligned} p_z^{(i)} - p_z^{(1)} &= E'/c;\ p_x^{(1)} = E\vartheta/c; \\ |\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}|^2 &= (1/c^2)(E'^2 + E^2\vartheta^2). \end{aligned} \tag{2.12}$$

Further

$$|\boldsymbol{p}^{(i)} - p^{(1)}|^2 - (1/c^2)(E^i - E^{(1)})^2 \backsimeq (1/c^2)((E'^2/\epsilon^2) + E^2\delta^2) \tag{2.13}$$

and

$$J^{(1)} = \frac{\partial(p_x^{(1)}, p_y^{(1)}, p_z^{(1)})}{\partial(E, \vartheta, \phi)} = \frac{p^{(1)}E^{(1)}}{c^2} \sin\vartheta \sim m^2\epsilon^2 c \sin\vartheta. \tag{2.14}$$

We start with the formulae (1.45), (1.46):

$$\begin{aligned} V_{p^{(1)}nl} = \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \tilde{u}_{nl}(r_2) \exp\{(2\pi i/h)[([\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}]\cdot\boldsymbol{r}_1) \\ - 1/c(E^{(i)} - E^{(1)})r]\}[(1 - (h/2\pi imc)\partial/\partial z)u_0(r_2)]/r. \end{aligned}$$

In all cases the matrix integral may be written as the product of two integrals if we replace throughout

$$\boldsymbol{r}_1 = \boldsymbol{r}_2 + \boldsymbol{r}_{12}.$$

Thus

$$V_{p^{(1)}nl} = \rho_{12}(p^{(1)})\rho_2(n, l, p^{(1)}) \tag{2.15}$$

where

$$\rho_{12}(p^{(1)}) = \int (d\boldsymbol{r}/r) \exp\{(2\pi i/h)[([\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}]\cdot\boldsymbol{r}) - (1/c)(E^{(i)} - E^{(1)})r]\} \tag{2.16}$$

and

$$\rho_2(n, l, p^{(1)}) = \int d\boldsymbol{r}\tilde{u}_{nl} \exp\left\{\frac{2\pi i}{h}([\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}]\cdot\boldsymbol{r})\right\}[1 - (h/2\pi imc)\partial/\partial z]u_0. \tag{2.17}$$

The former integral may be evaluated at once, and gives just, by (2.14)

$$\rho_{12}(p^{(1)}) = (h^2c^2/\pi)1/((E'^2/\epsilon^2) + E^2\vartheta^2). \tag{2.18}$$

The second integral may be considerably simplified, for it may be replaced by

$$\rho_2 = \int d\boldsymbol{r}\tilde{u}_{nl}e^{-i\gamma x}u_0;\ \text{with}\ \gamma = 2\pi mc\epsilon \sin\vartheta/h. \tag{2.19}$$

We may see this in the following way. We may expand

$$\exp\{(2\pi i/h)([\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}]\cdot\boldsymbol{r})\}$$

in powers of ϑ:

$$e^{(2\pi i/hc)\cdot E'z}\left\{1 - \frac{2\pi i}{hc}E\vartheta_x - \frac{2\pi^2}{h^2c^2}E^2\vartheta^2\cdots\right\}. \tag{2.20}$$

In all terms but the first the factor

$$e^{(2\pi i/hc)E'z}[1 - (h/2\pi imc)(\partial/\partial z)] \tag{2.21}$$

gives a correction of the order, by (2.12), E'/mc^2 which is small of the order $(v/c)^2$ for the atomic electron, and negligible. If we neglect (2.21) in the first term the integral vanishes; and we have to show that here, too, this neglect is justified. But now the integral

$$\int d\boldsymbol{r}\tilde{u}_{nl}e^{(2\pi i/hc)E'z}[1 - (h/2\pi imc)\partial/\partial z]u_0 \tag{2.22}$$

is just the matrix component of a perturbation produced by a field of potential

$$A_x = A_y = 0;\ A_z = -e^{(2\pi i/hc)E'(z-ct)};\ \phi = e^{(2\pi i/h)E'(z-ct)}. \tag{2.23}$$

These potentials may be derived for the scalar

$$\Lambda = -h/2\pi i e^{(2\pi i/hc)E'(z-ct)} \tag{2.24}$$

by differentiation $\boldsymbol{A} = \text{grad}\,\Lambda;\ \phi = -\partial\Lambda/\partial t$. They thus correspond to no field;

and the corresponding matrix element vanishes. (This may be verified by direct calculation as far as the dipole moment of the atom is concerned.)

The transition probability is thus given by

$$\sigma_{nl} = (e^4h^2c^2/\pi)\int d\gamma^2 \,|\, \rho_2(n,\,l,\,\gamma)\,|\,^2[1/(\gamma^2+\eta^2)^2] \tag{2.25}$$

with $\eta = 2\pi E'/hc\epsilon$ and $\rho_2(n,l,\gamma) = \int d\boldsymbol{r}\,\tilde{u}_{ne}e^{-i\gamma x}u_0$. This expression may, of course, be evaluated when we know the u's. We shall not need this evaluation to calculate the energy loss of the primary; but we do need it to compute the probability of ionization; and it is instructive to have the explicit expressions. We shall consider the very simplest case of an electron bound in the normal state of a hydrogen-like atom, with an effective nuclear charge z; we shall calculate σ for transitions to the continuum. We need to introduce the length

$$1/b = h^2/4\pi^2mze^2 \tag{2.26}$$

and the quantum number

$$n = 1/bk = 2\pi e^2z/hv;\; k = 2\pi mv/h;\, v = (2E/m)^{1/2} = [(2/m)(E'-I)]^{1/2}; \tag{2.27}$$

which measures the energy of the secondary electron. If we now use parabolic coordinates with the pole along the x axis, and use the well-known wave functions for the hydrogen-like atom in such coordinates, we can evaluate not only ρ_2, but the sum (integral) of $|\rho_2|^2$ over all states l of the same energy. We thus find

$$\sum_l |\,\rho_2(n,\,l,\,\gamma)\,|\,^2 = \frac{z^{10}\pi^2mb^6}{h^2}\,\frac{\exp\left[-\,2ntg^{-1}2kb/(b^2+\gamma^2-k^2)\right]}{1-e^{-2\pi n}}\,\frac{\gamma^2[\gamma^2+(b^2+k^2/3]}{\{[b^2+(\gamma-k)^2][b^2+(\gamma+k)^2]\}^3}. \tag{2.28}$$

For very small energy losses

$$E' \sim I \tag{2.29}$$

the integrand $G(\gamma)$ of the expression for the transition probability decreases with increasing γ, and becomes very large for $\gamma\rightarrow 0$. For small γ $G(\gamma)\sim\gamma^2/(\gamma^2+\eta^2)^2$ whereas for larger $\gamma\gg b$; $G(\gamma)\sim\gamma^{-12}$. Thus the transition probability begins to fall off more rapidly with increasing angle of deflection when the angle grows of the order

$$\bar{\vartheta} = hb/2\pi mc\epsilon. \tag{2.30}$$

This is the result quoted before (2.11) in connection with the semi-classical calculation. In this earlier calculation we neglected higher powers of γ higher moments of the atomic electron—and thus replaced $e^{-i\gamma x}$ by $1-i\gamma x$. And this reduced expression becomes wrong when $\gamma\sim b$, $\vartheta\sim\bar{\vartheta}$.

For energy losses large compared to the ionizing potential the value of G for $\gamma\rightarrow 0$ grows very small, and the integral over γ comes entirely from the region $\gamma^2 \sim k^2$; with neglect of higher powers of I/E' the integral may then be evaluated, and gives just

$$\sigma dE' = 2\pi e^4 dE'/mc^2 E'^2. \tag{1.40a}$$

This agrees with our result (1.40) for the energy loss to a free electron for $E' \ll mc^2$.

We may use (2.28) to compute the cross section for ionization. For (1.40) gives a negligible contribution for high energy losses, and we do not need here to supplement it with the result (1.40). We so find, carrying out approximately the quadratures over γ and E for the cross section for ionization

$$\sigma_{\text{ion}} = 0.29(2\pi e^4/mc^2 I) \ln (mc^2\epsilon^2/0.024I). \tag{2.31}$$

The nonrelativistic result is, for this case of a hydrogen like[3] atom

$$\sigma_{\text{ion}} = 0.29(2\pi e^4/mv^2 I) \ln (mv^2/0.024I) \tag{2.32}$$

where v is the velocity of the primary. The two differ by $(1.2\pi e^4/mc^2)\ln\epsilon$ for $v \rightarrow c$. The constant 1.2 is here computed only for the simple atom which we have considered; it depends upon the f-values for the atomic electron, and no universal closed formula can be given for it. We may get a somewhat more general result by comparing our formula with the nonrelativistic one. For (2.25) differs from the corresponding nonrelativistic formula in three points: (1) In the nonrelativistic formula c is to be replaced throughout by v. (2) In the nonrelativistic formula the quantity η in the denominator of (2.25) is to be replaced by $\eta' = 2\pi E'/hv$. (3) The upper limits of integration for γ and E' are greater in our formula. This third difference is not essential in computing the probability of ionization, since high energy loss impacts contribute relatively little to this probability. Now the nonrelativistic formula may always be written[3]

$$\sigma_{\text{ion}} = k_1(2\pi e^4/mv^2) \ln (mv^2/k_2 I). \tag{2.33}$$

By (1) this becomes

$$k_1(2\pi e^4/mc^2 I) \ln (mc^2/k_2 I), \tag{2.34}$$

and (2) may be taken into account in the following way. For the range $0 \leqq \gamma \leqq \eta'$, γ/b is small, and we may write

$$\int d\boldsymbol{r}\tilde{u}_{nl}e^{-i\gamma x}u_0 \sim -\, i\gamma x_{nl,0}, \quad x_{nl,0} = \int \tilde{u}_{nl} x u_0 d\boldsymbol{r}.$$

Thus the term to be added to the nonrelativistic cross section is

$$\frac{16\pi^3 e^4}{h^2c^2}\int_\theta^\infty dE \sum_l \left| x_{El,0} \right|^2 \ln \frac{\eta'^2}{\eta^2}\cdot \tag{2.35}$$

But the nonrelativistic cross section (2.34) itself may with $v = c$ be written

$$\frac{16\pi^3 e^4}{h^2c^2}\int_0^\infty dE \sum_l \left| x_{El,0} \right|^2 \cdot \ln \frac{mc^2}{k^2 I}$$

i.e., $k_1=(8\pi^2 mI/h^2)\cdot\int_0^\infty dE\sum_l |x_{El0}|^2$. Thus our result is

$$\sigma_{\text{ion}} = k_1(2\pi e^4/mc^2 I)\ \ln mc^2\epsilon^2/k_2 I. \tag{2.36}$$

This shows that the number of ions produced by a fast electron increases as its velocity approaches that of light. The ionizing power passes through a minimum for $\epsilon\sim 3$; the increase is very slow; if we use the value of k_2 given by (2.31) for a hydrogen-like atom; and if for air we take I to be about fifty volts, then the number of ions is double the minimum value for electrons of $\epsilon\sim 10^3$, or energy of the order of 10^9 volts.

To compute the mean energy loss, and thus the range, of the fast electron, we might again use the hydrogen-like atomic model, and supplement (2.28) with the corresponding expression for the probability of excitation. But here, too, it is simpler to compare our expression (2.25) with the corresponding non-relativistic one. It is simpler still to use the sum theorem of Bethe[3]

$$(8\pi^2 m/h^2\gamma^2)\sum_{nl}(E_n - E_0)\left|\int d\boldsymbol{r}\tilde{u}_{nl}e^{-i\gamma x}u_0\right|^2 = 1 \tag{2.37}$$

directly. If we apply this directly, we get for the mean energy loss

$$\delta E' = \frac{2\pi e^4}{mc^2}\int_0 \frac{d\gamma^2}{\gamma^2+\tilde{\eta}^2} \tag{2.38}$$

where $\tilde{\eta}=(2\pi/hc\epsilon)W$, and $W=k_3 I$ is an appropriately chosen mean value of the energy transferred per impact; and where the integration over γ is to be taken up to the maximum value permitted by the conservation laws. We have here, however, to consider that the method used in the derivation of(2.25) is not applicable for large energy losses and large values of γ, since it leads to (1.40a) in place of the correct (1.40). It is easy to correct for this, because of the fact that when

$$E' \gg I \tag{2.39}$$

we may use the calculations of §I for free electrons. Since when (2.39) is satisfied the conservation laws are fulfilled, we know that in the range $I\ll E'\ll mc^2$ we may set

$$E' = h^2\gamma^2/8\pi^2 m. \tag{2.40}$$

Thus if we integrate (2.38) up to a γ value corresponding to an energy loss $\overline{E}$:

$$I \ll \overline{E} \ll mc^2;\ \tilde{\gamma}^2 = 8\pi^2 m\overline{E}/h^2 \tag{2.41}$$

we get

$$\delta E' = (2\pi e^4/mc^2)\ \ln 8\pi^2 m\overline{E}/h^2\tilde{\eta}^2. \tag{2.42}$$

And we may add to this the mean energy loss given by (1.40) for impacts involving an energy transfer $E'\geqq\overline{E}\gg I$. This gives for large ϵ

$$\delta E'' = \int_{\bar{E}}^{E_m} E'dE'\,\frac{E^4+(E-E')^4+E'^4}{2E^2(E-E')^2E'^2} \sim \frac{2\pi e^4}{mc^2}\left[\ln\frac{mc^2}{\overline{E}} + \ln\epsilon + 9/8 - 2\ln 2\right] \tag{2.43}$$

Thus for the total energy loss

$$\delta E = \delta E' + \delta E''$$
$$= \frac{2\pi e^4}{mc^2}\left[2 \ln \frac{mc^2}{W} + 3 \ln \epsilon + 0.43\right] \tag{2.44}$$

which is our result for the mean energy loss.

The direct calculation with the model of a hydrogen-like atom gives (2.44) with the constant

$$k_3 = 1| \cdot |1; W = 1.1I. \tag{2.45}$$

This result gives just *half* the energy loss we should expect from Bohr's classical formula;[2] it gives twice the range. When ϵ is large we get for the range R of the particle in a gas containing N electrons per cc

$$R^{-1} = (4\pi e^4 N/m^2c^4\epsilon)\left[\ln mc^2/W + (3/2) \ln \epsilon\right] \tag{2.46}$$

where now W is a properly taken average energy of the order of the mean ionization potential. As ϵ grows large compared with mc^2/W this approaches

$$R^{-1} \rightarrow (6\pi e^4 N)/(m^2c^4) \ln \epsilon/\epsilon. \tag{2.47}$$

Note: We get an upper limit for the range if we set

$$\partial E'' = 0,$$

since this means that no large energy losses occur, and since we know that the neglects of our theory (radiative forces) can have no sensible effect upon the small energy losses. With $\delta E^{11} = 0$ we get asymptotically

$$R^{-1} \rightarrow 4\pi e^4 N/m^2c^4 \ln \epsilon/\epsilon,$$

which is four times the distance traveled according to the Klein-Nishina formula, by a gamma-ray before its first Compton encounter. This range (2.47) is just one-sixth the distance which (according to the Klein-Nishina formula) a quantum of energy (ϵmc^2) travels, on the average, before its first Compton encounter. Thus even for the greatest energies a gamma-ray is, according to theoretical results at present available, more penetrating than an electron of the same energy. It should be remarked that in the application of these formulae nuclear electrons may not, presumably, be neglected; they should be included in counting the total number (N) of electrons per cc. But unless ϵ is very large, the contribution of these electrons to the stopping power will be smaller than that of the extranuclear electrons, since the binding energy of the former must be of the order of a few million volts. It seems at present impossible to extend our calculations to the case of electrons bound in nuclei, because of our ignorance of the nature of that binding. But insofar as wave mechanics is applicable to such particles, (2.46) should give a reliable estimate of their effect on a fast electron.

III. Impacts of the Magnetic Neutron

We want now to study in a little more detail the expression which we have derived for the impact of a magnetic neutron with a free electron. The cross section for such encounters is given in (1.67) and the somewhat complicated quantities occurring in this expression are given in (1.69). We shall consider first and chiefly the case that the neutron has a velocity small compared with that of light; in this case it will be unable to give the electron a very high velocity, and we may greatly simplify (1.67) by neglecting higher powers of v/c for both particles. The complicated summation of the square of the numerators of (1.67) then reduces to

$$N = 2|T_{++}{}^4|^2 + 2|T_{+-}{}^4|^2 \tag{3.1}$$

and we find from (1.72)

$$N = (1/2M^2c^2)\{p'^4 + p'^2(4r^2 - 8r\Delta r + 2(\Delta r)^2] + (\Delta r)^4\}. \tag{3.2}$$

Further the denomination of (1.67) becomes

$$|\boldsymbol{p}^{(i)} - \boldsymbol{p}^{(1)}|^2 = (\Delta r)^2 + p'^2. \tag{3.3}$$

If we introduce for the ratio of the mass m of the secondary to that of the neutron

$$\lambda = m/M \tag{3.4}$$

and the further abbreviations

$$\alpha = \cos\vartheta,\ \beta = \sin\vartheta,\ g = [\alpha + (\lambda^2 - \beta^2)^{1/2}]/(1+\lambda), \tag{3.5}$$

we find

$$\sigma d\alpha = \frac{2\pi^3 e^2\mu^2}{h^2c^2} g d\alpha \frac{\{g^4\beta^4 + g^2\beta^2[4 - 8(1-g\alpha) + 2(1-g\alpha)^2] + (1-g\alpha)^4\}}{(1 - 2g\alpha + g^2)^2}\cdot \tag{3.6}$$

The energy transferred is

$$E' = (E/\lambda)(1 - 2g\alpha + g^2). \tag{3.7}$$

Several points are at once clear. Since for $\alpha \rightarrow 1$, $g \rightarrow 1$ the cross section behaves for small angles like

$$\sigma d\alpha \sim d\alpha/(1-\alpha) \sim d\beta^2/\beta^2 \tag{3.8}$$

whereas in the case of the Coulomb field it behaved like

$$d\alpha/(1\alpha)^2 \sim d\beta^2/\beta^4. \tag{3.9}$$

The mean energy transferred may be computed without considering the binding of the secondary, since $\delta E = \int \sigma E' d\alpha$ converges. Further, the cross section is independent of the velocity or energy of the neutron, and the mean energy loss is directly proportional to the initial energy; finally the cross section is proportional to the square of the magnetic moment of the neutron.

We shall reduce (3.6) for the case that the neutron has a mass equal to that of the secondary, $\lambda = 1$, or much smaller, $\lambda \gg 1$ or much larger $\lambda \ll 1$.

$\lambda = 1$. Here $g = \alpha$,

$$\sigma d\alpha = \frac{2\pi^3 e^2 \mu^2 \alpha}{h^2 c^2 \beta^4} \{\alpha^4 \beta^4 + 2\alpha^2 \beta^2 (2 - 4\beta^2 - \beta^4) + \beta^8\}. \qquad (3.10)$$

The quadrature over the angle of deflection gives for the energy loss

$$\delta E = \int \sigma E' d\alpha = (7/96)\sigma_0 E \qquad (3.11)$$

where $\sigma_0 = 16\pi^3 e^2 \mu^2 / h^2 c^2$. This cross section σ_0 may be written

$$\sigma_0 = \pi e^4 n^2 / m^2 c^4, \text{ with } n = \mu/\mu_B; \ \mu = eh/4\pi mc \cdot n \qquad (3.12)$$

where n is the magnetic moment of the neutron measured in Bohr magnetons. This quantity n is presumably quite small, so that σ_0 is very small indeed. The expression for the total number of impacts

$$\sigma = \int \sigma d\alpha \qquad (3.13)$$

does not converge in the small angles. The preponderance of small energy losses is not nearly so marked here as for the impacts of an electron; and it is easy to show, by repeating the familiar nonrelativistic calculations for the impact of a particle with a bound electron, that the total number of impacts in which an energy of the order of a few times the binding energy or less in transferred is finite and independent of the primary energy and corresponds to a cross section of the order of σ_0. The total cross section thus turns out to be

$$\sigma = \int \sigma d\alpha \sim \sigma_0 \{\tfrac{1}{4} \ln (E/W) - 7/16\} \qquad (3.14)$$

where W is of the order of the ionizing energy. The mean energy transferred per impact is thus

$$\delta E/\sigma \sim (E/24) \ln E/W. \qquad (3.15)$$

Such a magnetic neutron, quite apart from the great infrequency of its impacts, a factor $In^2/4mc^2$ smaller cross section than an electron—will never produce ion tracks in a cloud chamber, since it tends to lose an appreciable fraction of its energy, and suffer an appreciable deflection at every impact.

$\lambda \gg 1$. For this case we find

$$\sigma \sim (\sigma_0/4) \ln 2E/\lambda W \qquad (3.16)$$

$$\delta E = (19/96)\sigma_0 E/\lambda \qquad (3.17)$$

and for $\lambda \ll 1$:

$$\sigma = (\sigma_0/4) \ln E\lambda/W \qquad (3.18)$$

$$\delta E = \sigma_0 E\lambda/8, \qquad (3.19)$$

In both these cases impacts are relatively rare; in both, the mean energy loss is necessarily smaller than for equal masses; only for a heavy neutron are impacts without large deflection possible. If such a neutron as this had a large magnetic moment, large compared to the Bohr magneton, it could produce recognizable cloud chamber tracks; but values of n large enough to give this seem *a priori* extremely improbable.

The formulae we have given apply equally—especially to the impacts of a magnetic neutron with a proton or nucleus. They do not account very well for the phenomena observed in the penetrating radiation from Be bombarded with alpha-particles. In the first place, such a magnetic neutron would have more impacts with highly charged nuclei than with protons, whereas the observations show that this is not so; in the second place the number of impacts would depend so little on the energy of the neutron, so that it would be hard to account for the observed inhomogeneous absorption of the radiation. Finally the distance between impacts would be enormous; with $n=10^{-3}$ and $\lambda=1$, a neutron of 5.10^6 volts would travel$\sim$100 km before ejecting a visible proton from paraffin.

One might suppose that in the general case of velocities not small compared to that of light, different and more satisfactory results might be obtained. We have investigated this case only for $\lambda=1$, in which case the conservation laws give simple algebraic relations between angle of deflection and final momenta and energy. For this case the total cross section increases with ϵ, where again $E=\epsilon Mc^2$ is the initial energy of the neutron, and is of the order.

$$\sigma_0\epsilon^2. \tag{3.20}$$

In this case $\epsilon\gg 1$, the preponderance of small deflections and relatively small energy losses is still less marked than in the case of low velocity magnetic neutrons. When

$$\epsilon > \ln E/W$$

the electron tends to lose a large part of its energy in each encounter it makes. Such a neutron could not produce cloud chamber tracks; and since it is certain on energetic grounds that the radiation from beryllium does not consist of such neutrons, we have not thought it desirable to give further details in the evaluation of (1.67).

We believe that these computations show that there is no experimental evidence for the existence of a particle like the magnetic neutron.